ARBITRAGE THEOREM AND ITS APPLICATIONS

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SUMMARY

In my article I describe the concept of financial rate of return and the value of return in a very simple model first. Then as generalisation of the model we take an experiment, which has n possible outcomes. We have the same m kind betting possibilities for each outcome. The financial rate of return is known for each outcome and betting possibility. We define the concept of arbitrage (the possibility of sure winning), and we are looking for the answer how to characterise the arbitrage exemption. What is the guarantee, that any betting terms cannot be given for which the winning is sure for each outcome? For this question the answer is given in the arbitrage theorem, which is one of the alternatives of the well-known Farkas theory. In the second part of the article I demonstrate some applications of the theorem. I apply it for a classical betting problem first, then for an option pricing in more details. The applications for the one-period binomial and trinomial, and the more-period binomial option pricing will also be made known.

1. INTRODUCTION

Consider the following simple model. A bookmaker takes a bet and gets a certain amount. Let x be the bet and the win z. The ratio $r=z/x$ is called return rate and denoted by r. If the bet is x, then the return value is $z = rx$.

The return rate is the return value of the unit bet. The bet, the return rate and the return value can be an arbitrary real number. If e.g. the bet is 3, and the return rate is 2, then bookmaker wins 6, (if $x = 3$, $r = 2$, then $z=6$). If e.g. $x = 5$ and $r = -3$, then $z = -15$, then bookmaker loses 15.

Now we generalize our simple model. In the new model let m be the number of the wagers. Let $r_1, r_2, ..., r_m$ be the return rate of wagers. If $x_1, x_2, ..., x_m$ are the bets on wagers, then the return value of the wagers is

$$z = \sum_{i-1}^{m} x_i r_i \quad .$$

If the value of z is positive, then the bookmaker wins, if z is negative, then the bookmaker loses.

In a further generalization of the model, consider an experiment where the number of possible outcomes is n. We use the same wagers (W) for all outcomes (O). We assume that the return rates for all outcomes are known. The bets can then be denoted by vector **x**, the return rates can be denoted by matrix **R**, where the entry r_{ij} stands for.

x

R

The following scheme shows our data

		$O_1 \cdots$	$O_j \cdots$	O_n
x_1	W_1	$r_{11} \cdots$	$r_{1j} \cdots$	r_{1n}
:				
x_i	W_i	$r_{i1} \cdots$	$r_{ij} \cdots$	r_{in}
:				
x_m	W_m	$r_{m1} \cdots$	$r_{mj} \cdots$	r_{mn}

If the outcome of experiment is O_j, then the return value for some betting vector **x** is

$$z_j = \sum_{i=1}^{m} x_i r_{ij.} \quad .$$

If there is such a betting vector **x** that

$$z_j > 0, \ (j= 1, ..., n)$$

this means that we have a sure win for each outcome of the experiment. This is called arbitrage or arbitrage opportunity (arbitrage = sure win opportunity).

2. Arbitrage Theorem

The arbitrage can also be written in matrix form, that is if there is such a betting vector **x** that

$$\mathbf{XR} > 0,$$

then arbitrage occurs.

In the following we give a characterization of the so-called arbitrage-free. What is the guarantee that doesn't exist such a betting vector **x** which leads to sure win opportunity? The Farkas theorem gives the answer to this question.

Farkas theorem:

The system

$$\mathbf{XR} > 0,$$

has no solution if and only if the system

$$\mathbf{Rp} = 0$$
$$\mathbf{p} \geq 0$$
$$\mathbf{p} \neq 0$$

has solution.

This theorem is known as Gondan theorem too, but this is an other form of the original Farkas theorem.
In the original Farkas theorem the two systems are $\mathbf{Ax=b}$, $\mathbf{x \geq 0}$ and $\mathbf{yA \leq 0}$, $\mathbf{yb>0}$. J. Farkas published this famous theorem in 1902 and he applied it for the axiomatization of the analytical mechanics. The Farkas type theorems play an important role in the field of optimization. This is one of the most quoted theorem in the topic of optimization.
If we consider vector $\mathbf{p}=(p_1, p_2, \ldots, p_n)$ in the second system as a probability (random) vector of the outcomes $\{O_1, O_2, \ldots O_n\}$ then vector **Rp** can be interpreted according to the following: The i-th element of vector **Rp**

$$\sum_{j=1}^{n} r_{ij} p_j$$

is the expected value of the returns value of the i-th wager. According to the Farkas theorem arbitrage doesn't occur if and only if the expected value of the return values is zero for all wagers. To summarize we point out that the arbitrage can be formulated in the following way.

Arbitrage theorem:
Exactly one of the following statements is true:
a) there exists a probability vector $\mathbf{p}=(p_1, p_2, \ldots, p_n)$ for which

$$\sum_{j=1}^{n} r_{ij} p_j = 0$$

for all $i=1, 2, \ldots, m$,
b) there exists a betting vector $\mathbf{x}=(x_1, x_2, \ldots, x_m)$ for which

$$\sum_{i=1}^{m} x_i r_{ij} > 0$$

for all $j=1, 2, \ldots, n$.

Proof of the theorem:
The arbitrage theorem can be proved in several ways. Here we prove it by means of the duality of linear programming. Consider the standard primal and dual linear programming problems:

Primal	Dual
$\mathbf{Az} = \mathbf{b}$ $\mathbf{z} \geq 0$ $\mathbf{cz} \rightarrow$ min!	$\mathbf{yA} \leq \mathbf{c}$ $\mathbf{yb} \rightarrow$ max!

According to the duality theorem of the linear programming if the primal and the dual problems have feasible solutions then both problems have optimal solutions and the minimal value of the primal objective function is equal to the maximal value of the dual objective function.
Consider the following linear programming problem.

$$\sum_{i=1}^{m} x_i r_{ij} \geq x_{m+1} \quad (1)$$

$$x_{m+1} \rightarrow max$$

According to the condition of the problem we would like to reach at least an amount x_{m+1} for all outcomes and beside we want that this amount should be maximal. This problem can be transformed to the standard dual linear programming problem and we can write the primal problem as follows:

$$\sum_{j=1}^{n} p_j = 1, \quad (2)$$

$$p_j \geq 0, \quad j = 1, 2, \ldots, n$$
$$0 \rightarrow \min$$

Note that the condition of problem (2) is the same as in the arbitrage theorem. It can be easily observed that the problem (1) has feasible solution (e.g. $\mathbf{x}=\mathbf{0}$ and $x_{m+1}=0$). We distinguish two cases according that the problem (2) has or hasn't got any solution.
If the problem (2) has feasible solution (there exists a probability vector) then according to the duality theorem both problems have optimal solutions, the optimal value is zero. So $x_{m+1}=0$ means that there is no sure win opportunity.
If the problem (2) has no feasible solution (there doesn't exist a probability vector) then according to the duality theorem there isn't any optimal solution and the objective function of problem (1) is not bounded from above. It means that x_{m+1} can be positive. In this case there is sure win for all outcomes, so there is arbitrage. The arbitrage theorem has been proved.
The arbitrage theorem has a weak form, which gives a connection for the sure not-lose opportunity instead of the sure win.

Weak arbitrage theorem:

Exactly one of the following statements is true:

a) there exists a probability vector $\boldsymbol{p}=(p_1, p_2, ..., p_n)$, all of whose components are positive for which

$$\sum_{j=1}^{n} r_{ij} p_j = 0$$

for all $i = 1, 2, ..., m$,

b) there exists a betting vector $\boldsymbol{x}=(x_1, x_2, ..., x_m)$ for which

$$\sum_{i=1}^{m} x_i r_{ij} \geq 0$$

for all $j= 1, 2, ..., n$, but for at least one index the strict inequality holds.

3. Applications

3.1. Classical odds

Consider an experiment with n possible outcomes on which we can bet. The odds can be given with the scheme e.g. "3 to 1", this means that the bet is 1, the return value is 3 if the outcome of the experiment is favourable for us in the other case the bet is lost. Let our bet for the i-th outcome of the experiment "o_i to 1". In this problem the matrix **R** is quadratic, which has the following entries:

$$r_{ij} = 0_i, \quad i = 1, 2, ..., n,$$
$$r_{ij} = -1, \quad i \neq j.$$

According to the arbitrage theorem we have possibility for the sure win if and only if there exists a probability vector $\boldsymbol{p}=(p_1, p_2, ..., p_n)$ for which

$$0 = \sum_{j=1}^{n} r_{ij} p_j = o_i p_i + (-1)(1-p_i), (i = 1,2,...,n),$$

from which we obtain that

$$p_i = \frac{1}{1+o_i}, (i = 1,2,...,n),$$

Since p_i is probability, the p_i must sum to 1. If

$$\sum_{i=1}^{n} \frac{1}{1+o_i} \neq 1,$$

then we can give such a bet for which we have sure win opportunity independently of the outcome of the experiment. It can be easily shown that the bets

$$x_i = \frac{\frac{1}{1+o_i}}{1-\sum_{k=1}^{n}\frac{1}{1+o_k}}, i = 1,2,...,n$$

yield sure win and the gain is 1 for all outcomes. If e.g. there are three outcomes and for these the odds are "1 to 1", "2 to 1", "3 to 1", then we have sure win, because

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12} \neq 1 .$$

If e.g. the bets are -1, -0.7, -0.5, then our gains are 0.2, 0.1, 0.2 . If the bets are - 6, -4, -3, then the gain is 1 for all outcomes.

3.2. Option pricing

Consider a European call option, we want to determine its price C. The call option is a right (not the obligation) to buy a stock for a given price (exercise or strike price) at a given future time (expiration date). In this section we use the following notations:

- S : initial (present) price of the stock,
- K : exercise price of the call option,
- T : expiration time (year),
- r : nominal risk-free interest rate per year (compounded continuously),
- S_T : stock price at the expiration time.

First we compute the value of the call option at the expiration time. If the stock price (S_T) at the expiration time exceeds the exercise price (K) of the call option, then the owner of the option will exercise the option and buy the stock for price K. The value of the call option is $S_T - K$. If the stock price (S_T) at the expiration time doesn't exceed the exercise price (K) of the call option then the owner of the option wouldn't exercise the option. The option is worthless, so its value is zero. We can handle both cases together with the help of maximum function or notation of positive cutting. The value of the call option at the expiration time is

$$\max(S_T - K, 0),$$

or with notation of the so-called positive cutting:

$$(S_T - K, 0)^+.$$

3.2.1. Single period binomial model

Suppose that the initial stock price is S. At the end of the period the stock price will either be Su or Sd. We assume that

$$d < 1+e^{rT} < u$$

The stock price at the expiration time is

$$S_T = \begin{cases} Su \\ Sd \end{cases} .$$

Here we use the previous terminology (experiment, outcome, etc.). In this problem the experiment has two outcomes, which are stock prices at the expiration time. Now consider a portfolio, which consists of a stock and a call option. So we have two wagers: buying (or selling) the stock respectively buying (or selling) the call option. What is the matrix **R** in our new model? The entry r_{ij} (i,j=1,2) means the present return value in the case of buying of a stock and buying of an option. As known, present value = value times e^{-rT}.

r_{11}: at the expiration time the value of the stock is Su, the return value is its present value reduced by the cost of buying stock.

r_{12}: at the expiration time the value of the stock is Sd, the return value is its present value reduced by the cost of buying stock.

r_{21}: at the expiration time the value of the call option is $(Su–K)^+$, the return value is its present value reduced by the cost of buying option.

r_{22}: at the expiration time the value of the call option is $(Sd–K)^+$, the return value is its present value reduced by the cost of buying option.

The return matrix **R** can be written in the following way

	Increasing stock price	Decreasing stock price
Stock	Sue^{-rT}-S	Sde^{-rT}-S
Call option	$(Su–K)^+$-C	$(Sd–K)^+$-C

We want to determine the price of call option (C) in such a way that there is not sure win. There is no such a portfolio for which there is sure win independently from the stock price. The answer for this is given by the arbitrage theorem. Let p be the probability that the stock price increases. Let (1-p) be the probability that the stock price decreases. According to the arbitrage theorem there is no arbitrage opportunity is not if there is a probability p that the expected return of the stock and the call option is zero, that is

$$p(Sue^{-rt}-S)+(1-p)(Sde^{-rT}-S)=0,$$

$$p\left[\ (Su-K)+-C\right]\ +(1-p)\left[(Sd–K)^+-C\right]=0$$

After solving this system of equations the option price becomes

$$C=e^{-rT}\left[p(Su–K)^+ +(1-p)(Sd–K)^+\right],$$

where the probability, that the stock price increases, is

$$p=\frac{e^{rT}-d}{u-d}\ .$$

To summarize we note that the price of the call option is the present value of the expected value of the option values at the expiration time.

If e.g. the initial stock price S=200, the factors u=1.1, d=0.9, the exercise price of the call option K=210, the expiration time T=0.5 year, the risk-free interest rate 12 %, so r=0.12, then p=0.8092 and C=7.621 . The price of the call option is 7.621.

3.2.2. Single period trinomial model

Consider a European call option. Let S be the initial price of stock. Suppose that the stock price will have three possible prices at the end of the period. Let u, v, d be the factors so the stock price will be Su, Sv respectively Sd. The price of the call option (C) can be determined in the following way.

In the trinomial model also consider a portfolio with stock and call option. The matrix **R** is the following:

	Su	Sv	Sd
Stock	Sue^{-rT}-S	Sve^{-rT}-S	Sde^{-rT}-S
Call option	$(Su–K)^+$-C	$(Sv–K)^+$-C	$(Sd–K)^+$-C

Let be the probabilities of the stock prices Su, Sv, Sd. According to the arbitrage theorem we obtain the following system of equations

$$p_1(Sue^{-rT}-S)+p_2(Sve^{-rT}-S)+p_3(Sde^{-rT}-S)=0$$

$$p_1\left[(Su–K)^+-C\right]+p_2\left[(Sv–K)^+-C\right]+p_3\left[(Sd–K)^+-C\right]=0$$

From the second equation

$$C=p_1(Su–K)^+ +p_2(Sv–K)^+ +p_3(Sd–K)^+,$$

from the first equation

$$p_1u^+ +p_2v+p_3d=e^{-rT}.$$

Considering that $p_1+p_2+p_3=1$, we obtain for the option price that

$$C=p_1\left((Su–K)^+-(Sd–K)^+\right)+p_2\left((Sv–K)^+-(Sd–K)^+\right)+(Sd–K)^+,$$

$$p_1(u-d)+p_2(v-d)=e^{-rT}-d,$$

where $p_1, p_2 \geq 0$, $p_1, p_2 \leq 1$. In the trinomial model the arbitrage-free option price is not unique.

If e.g. the initial stock price S=100, the three possible stock price at the expiration time Su=120, Sv=102, Sd=90, the exercise price of the call option K=105, the expiration time T=1 year, the risk-free interest rate 12 % (r=0.12), then

$$4.763 \leq C \leq 10.085.$$

If the price of the call option falls into the above interval then there is no arbitrage opportunity.

3.2.3. Multiperiod binomial model

Finally consider an option-pricing problem in which there are *n* periods. We divide the period time (*T*) *n* parts of the same size. Let *r* be the interest rate per year and the interest rate is the same in all periods, let *K* be the exercise price of the call option, let S_0 be the initial stock price. Let S_i be the stock price at the end of the *i*-th period (*i*=1,2,…,*n*). Suppose that the stock price either increases or decreases in all periods. The increasing and decreasing factors are *u* respectively *d*, where

$$d < 1+e^{r\frac{T}{n}} < u$$

Let X_i be a Bernoulli random variable, which characterizes the changing of the stock price at the end of the *i*-th period considering the stock price at the end of the (*i*-1)-th period:

$$X_i = \begin{cases} 1, if S_i = uS_{i-1} \\ 0, if S_i = dS_{i-1} \end{cases}$$

In our model the outcome of the experiment is the values of the random vector (X_1, X_2, ..., X_n). According to the arbitrage theorem there is not sure win if for these outcomes there exists such a probability that the expected value is zero. So there must be a set of probabilities

$$P(X_1=x_1, X_2=x_2, ..., X_n=x_n), \quad x_1=0.1, \quad i=1,2,...,n.$$

that makes all bets fair. Now consider the following type of bet. First choose a period e.g. *i*-th period (*i*=1,2,…,*n*) and to this period we choose an arbitrary vector which assumes 0 or 1 elements. This vector shows how to change the stock price until the *i*-th period. If $X_j=x_j$ for all *j*=1,2,…,*i-1*, then we choose the following strategy: We buy one unit stock and sell it back the next period. When we buy the stock in the (*i*-1)-th period, then its cost is S_{i-1}, when we sell it in the *i*-th period, then we get either amount uS_{i-1} or dS_{i-1}. The present value in the (*i*-1)-th period of these amounts can be obtained if we multiply it by $e^{-rT/n}$. The return value in the (*i*-1)-th period can be the following two values:

$$e^{-r\frac{T}{n}}uS_{i-1} - S_{i-1} \qquad or \qquad e^{-r\frac{T}{n}}dS_{i-1} - S_{i-1}$$

Let *q* be the probability that the stock is purchased, so

$$q=P(X_1=x_1, X_2=x_2, ..., X_{i-1}=x_{i-1}).$$

Let *p* be the probability (conditional probability) that the price of a purchased stock increases in the next period, that is

$$p=P(X_1=1 | X_1=x_1, X_2=x_2, ..., X_{i-1}=x_{i-1}).$$

The probability (1-*p*) means that the price of a purchased stock decreases in the next period. The expected return value at the (*i*-1)-th period can be computed in the following way

$$pq\left(e^{-t\frac{T}{n}}uS_{i-1} - S_{i-1}\right) + q(1-p)\left(e^{-t\frac{T}{n}}dS_{i-1} - S_{i-1}\right).$$

According to the arbitrage the above bet is arbitrage-free if this expected value is zero, that is

$$pq\left(e^{-t\frac{T}{n}}uS_{i-1} - S_{i-1}\right) + q(1-p)\left(e^{-t\frac{T}{n}}dS_{i-1} - S_{i-1}\right) = 0.$$

After reducing the equation we obtain that

$$pue^{-t\frac{T}{n}} + (1-p)de^{-t\frac{T}{n}} = 1,$$

from this, the probability *p* is

$$p = \frac{e^{r\frac{T}{n}} - d}{u-d},$$

We obtained that the only probability vector that results arbitrage-free for this bet is the following

$$P\left(X_i = 1 | X_i = x_i, X_2 = x_2, ..., X_{i-1}\right) = \frac{e^{r\frac{T}{n}} - d}{u-d}.$$

Since the vector is arbitrary this implies that the probability, that the stock price increases, is the same in all period and equal to the above conditional probability, that is

$$P(X_i = 1) = p = \frac{e^{r\frac{T}{n}} - d}{u-d}, i = 1,2,...,n.$$

The above implies that the random variables X_1, X_2,…, X_n are independent, all having the same distribution. The expected values $E(X_i)$ and the variances $Var(X_i)$ (*i=1,…,n*) are the following

$$E(X_i)=p,$$
$$Var(X_i)=p(1-p).$$

Let *Y* be a new random variable defined by

$$Y = \sum_{i=1}^{n} X_i, ,$$

that is the sum of random variable X_i, The random variable *Y* shows the number of the increasing of the stock price in the *n* period. The random variable (*n-Y*) shows the number of the decreasing of the stock price in the *n* period. This random variable has binomial distribution with the following expected value and variance

$$E(Y)=np,$$
$$Var(Y)=np(1-p).$$

At the end of the total period the stock price S_n is also a random variable, which can be expressed with random variable *Y* in the following way

$$S_n=S_0u^yd^{n-y},$$

The value of the call option at the expiration time is $(S_n\text{-}K)^+$, its present value at the beginning of the period is

$$e^{-rT}(S_n-K)^+$$

and its expected value will be the price of the call option C. Summarizing, the following option price results arbitrage-free

$$C=e^{-rT}E((S_0u^Yd^{n-Y})^+).$$

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Összefoglalás

Cikkemben először egy nagyon egyszerű modellben ismertetem a megtérülési ráta és a megtérülési érték fogalmakat. Majd a modell általánosításaként tekintsünk egy olyan kísérletet, amelynek n lehetséges kimenetele van. Mindegyik kimenetre ugyanaz az m féle fogadási lehetőségünk legyen. Ismert minden kimenetre és fogadási lehetőségre a megtérülési ráta. Definiáljuk az arbitrázs (biztos nyerés lehetősége) fogalmát, és arra keressük a választ, hogy az arbitrázsmentességet hogyan lehet karakterizálni. Mi garantálja azt, hogy nem lehet olyan fogadási tételeket megadni, amelynél minden kimenetre biztos a nyerés? Erre a választ az arbitrázs tételben adjuk meg, ami az ismert Farkas tétel egy változata. A cikk második részében a tétel néhány alkalmazását mutatom be. Elsőként egy klasszikus fogadási problémára, majd részletesebben az opcióárazásra alkalmazom a tételt. Ismertetésre kerül az egyperiódusos binomiális és trinomiális ill. a többperiódusos binomiális opcióárazásra történő alkalmazás.

Zusammenfassung

In meinem Artikel zuerst bespreche ich die Umschlagsrate und den Umschlagswert in einem einfachen Modell. Dann als Generalizierung des modells betrachten wir ein Experiment mit n Ausgaben. Alle Ausgaben haben den gleichen Wetten. Die Umschlagsraten sind bekannt für alle Ausgaben und Wetten. Wir definieren die Arbitrage, und wir suchen die Antwort für die Fragen: Was charakteriziert die Arbitrage-frei? What is die Garantie dazu, dass wir keinen Beten geben können, dass das Gewinnen für alle Ausgaben sicher ist? Die Arbitrage Satz gibt die Antwort für diese Fragen. Dieser Satz ist ein Version des Farkas Satzes. Im zweiten Teil des Artikels demonstrieren wir einige Anwendungen des Arbitrage Satzes. Erstens anwenden wir diesen Satz für eine classishe Wette, dann für die Optionspreisung.

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